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Approximation by Operators of Probabilistic Type

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This paper is concerned with sets $\mathfrak{W}$ of sequences $(W)_{n=1}^{\infty} \in \mathfrak{W}$ of positive linear operators which are of certain probabilistic type and act on certain function classes K. Necessary and sufficient conditions upon $\mathfrak{W}$ are determined such that each element $(W_n)_{n=1}^{\infty} \in \mathfrak{W}$ approximates U with a given order of approximation $\psi(n)$ and a given function class K, the limiting operator U being either the identity I or an operator connected with the normal distribution. The saturation problem in this setting is also solved, now in a form giving the order of saturation $\psi(n)$ such that convergence of $(W_n)_{n=1}^{\infty}$ towards U of order $o(\psi(n))$, $n \to \infty$, is impossible unless $W_n = I$, $n \in \mathbb{N}$, and there exists a non-trivial element $(W_n)_{n=1}^{\infty}$ which approximates U with order $\mathcal{O}(\psi(n))$.

1. Introduction

In the classical saturation problem, one considers a fixed approximation process $(U_n)_{n=1}^{\infty}$ and asks for its saturation class

$$S = \{f; \| U_n f - f \| = \mathcal{O}(\psi(n))\}, \tag{1.1}$$

where ψ is to be determined in such a way that $\| U_n f - f \| = o(\psi(n))$, $n \to \infty$, implies f to be trivial. Several mathematicians such as Zhuk and Natanson [9] consider what they call a converse problem of saturation theory: they start with a class of functions K and try to find *one* approximation process for which K is the saturation class.

In this paper we also consider a fixed function class K, but we are interested in *all* sequences of operators (of a certain probabilistic type) that approximate each $f \in K$ with a prescribed order of approximation ψ, not necessarily the order of saturation. Moreover, looking at (1.1), we consider convergence of $(U_n)_{n=1}^{\infty}$ to not merely to the identity operator I but also to some other operators, and, in contrast to (1.1), not (only) in the norm but even in the pointwise sense.

More exactly, let K be a fixed function class defined on some interval $J \subseteq \mathbb{R}$, and consider $\mathfrak{U}$, the set of sequences of positive linear operators $(U_n)_{n=1}^{\infty}$

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with $K \subset D(U_n)$, the domain of U_n, as well as some positive linear operator, namely the limiting operator U with $K \subset D(U)$. Given K, U, ψ, the problem is to determine the subclass $\mathfrak{U}_0 \subset \mathfrak{U}$ such that

$$|U_n f(x) - Uf(x)| = \mathcal{O}(\psi(n)) \qquad (\text{each } f \in K) \tag{1.2}$$

if and only if $(U_n)_{n=1}^{\infty} \in \mathfrak{U}_0$, where $x \in J$ is fixed and ψ describes the order of approximation, i.e., to find

$$\mathfrak{U}_0 = \{(U_n)_{n=1}^{\infty} \in \mathfrak{U};\ |U_n f(x) - Uf(x)| = \mathcal{O}(\psi(n)), \text{ each } f \in K\}. \tag{1.3}$$

$\mathfrak{U}_0$ is the saturation class in this setting, now called satiety class, if (i) $\mathfrak{U}_0$ is not the trivial class, i.e., we do not have, for all $f \in K$, $n \in \mathbb{N}$,

$$U_n f(x) = Uf(x), \tag{1.4}$$

but, $f \in K$ and

$$\text{(ii)} \quad |U_n f(x) - Uf(x)| = o(\psi(n)) \tag{1.5}$$

imply (1.4); in this case $\mathfrak{U}$ is said to be satiated with satiety order $\psi(n)$. Notice that the large "$\mathcal{O}$" and small "o" terms may depend on the fixed point $x \in J$, the operator U and the function $f \in K$. Of course, one could detach the matter from a fixed $x \in J$ and consider the problem for all $x \in J$, i.e., add "$x \in J$" in formulas (1.2)–(1.5), the problem nevertheless still being pointwise. Recall that assertions such as (1.2) or (1.5) for, e.g., the Szàsz–Mirakjan or γ operators do not hold uniformly, so that we have to consider them in the pointwise sense.

In the following, as indicated above, we do not consider the whole class $\mathfrak{U}$ but only a subclass $\mathfrak{W} \subset \mathfrak{U}$ such that the elements $(W_n)_{n=1}^{\infty} \in \mathfrak{W}$ are of the probabilistic form

$$\begin{aligned} W_n(\varphi(n))\, f(x) &= E(f \circ [\varphi(n)(X_1 + X_2 + \cdots + X_n)]) \\ &= E(f \circ \varphi(n)\, S_n) = \int_{-\infty}^{\infty} f(u)\, dF_{\varphi(n) S_n}(u) \qquad (\text{each } f \in K), \end{aligned} \tag{1.6}$$

where $(X_i)_{i=1}^{\infty}$ are real independent, identically distributed (i.i.d.) random variables (r.v.) which will be constructed in dependence upon $x \in J$, S_n is the nth partial sum of the r.v. X_i, and φ is a normalizing function $\varphi\colon \mathbb{N} \to \mathbb{R}$, tending to zero for $n \to \infty$; in the sequel we consider two concrete specializations of φ, namely $\varphi(n) = n^{-1}$ and $\varphi(n) = n^{-1/2}$. Examples of operators that belong to $\mathfrak{W}$ are the Bernstein, Szàsz–Mirakjan and Baskakov operators in the discrete case, and the Weierstrass and γ operators (in Feller's version) in the continuous case. The probabilistic structure of the operators guarantee at once that $W_n(\varphi(n))1 = 1$.

The precise formulations of the main results are to be found in Theorems 1, 2 and 3; they will be proved by elementary stochastic methods.

2. Notations and Preliminaries

Let $(X_i)_{i=1}^{\infty}$ be a sequence of real i.i.d.r.v., all distributed as a r.v. X, and assume the limiting r.v. Z to be φ-decomposable, i.e., for each $n \in \mathbb{N}$ there exist n independent r.v. $Z_{1,n}, \ldots, Z_{n,n}$ such that

$$F_Z = F_{\varphi(n)\sum_{i=1}^{n} Z_{i,n}},$$

where F_Y denotes the distribution function (d.f.) of a r.v. Y (the decomposability concept was introduced in [3] to prove general limit theorems in probability theory).

Further, let $C(J)$ denote the vector space of all real-valued continuous functions on an interval $J \subseteq \mathbb{R}$, and $C_b(J)$ the subset of all bounded and uniformly continuous functions on J. For $r, j \in \mathbb{P} := \mathbb{N} \cup \{0\}$ set

$$C_b^r(J) := \{f \in C(J); f', f'', \ldots, f^{(r-1)} \in C(J), f^{(r)} \in C_b(J)\},$$

and $f_j(x) := x^j$. Finally, let $]\alpha[$ be the greatest integer less than or equal to $\alpha > 0$, and $[\alpha]$ be the smallest integer greater than or equal to $\alpha > 0$. Two general inverse theorems on r.v. will be needed (see [6]).

THEOREM. *Let the rth moment $E(Z^r)$ be finite for some $r \in \mathbb{N}$.*

A. *If for some $\beta > 0$*

$$\int_{-\infty}^{\infty} f(u)\, d[F_{\varphi(n)S_n}(u) - F_Z(u)] = \mathcal{O}(n\varphi(n)^{\beta})$$

for each $f \in C_b^r(\mathbb{R})$, then

(i) $E(X^r) < \infty$,

(ii) $E(X^j) = E([Z/\varphi(1)]^j)$ $(0 \leqslant j \leqslant [\min(r, \beta)] - 1)$.

B. *If*

$$\int_{-\infty}^{\infty} f(u)\, d[F_{\varphi(n)S_n}(u) - F_Z(u)] = o(n\varphi(n)^r) \qquad (n \to \infty)$$

for each $f \in C_b^r(\mathbb{R})$, then

(i) $E(X^r) < \infty$,

(ii) $E(X^j) = E([Z/\varphi(1)]^j)$ $(0 \leqslant j \leqslant r)$.

In view of $C_b^r(\mathbb{R}) \subset C_b^r(J)$ one may replace $C_b^r(\mathbb{R})$ by $C_b^r(J)$ for some interval $J \subset \mathbb{R}$.

3. Theorems Connected with the Limiting Operator I

In this section we consider the case $\varphi(n) = 1/n$ and $U = I$, so that the operators U_n are of the form (cf. (1.6))

$$W_n(1/n)f(x) = E(f \circ S_n/n) \qquad (n \in \mathbb{N}). \tag{3.1}$$

THEOREM 1. *Let $x \in J$ be fixed.*

(a) *If*

$$|W_n(1/n)f(x) - f(x)| = o(1) \qquad (n \to \infty) \tag{3.2}$$

for each $f \in C_b^1(J)$, then

(i) $E(X) < \infty$,

(ii) $W_n(1/n)f_1(x) = f_1(x) = x \ (n \in \mathbb{N})$.

(b) *If*

$$|W_n(1/n)f(x) - f(x)| = \mathcal{O}(n^{-\alpha}) \tag{3.3}$$

for each $f \in C_b^2(J)$ and some $\alpha > 0$, then

(i) $E(X^2) < \infty$,

(ii) $W_n(1/n)f_1(x) = f_1(x) = x \ (n \in \mathbb{N})$.

(c) *If*

$$|W_n(1/n)f(x) - f(x)| = o(n^{-1}) \qquad (n \to \infty) \tag{3.4}$$

for each $f \in C_b^2(J)$, then

(i) $E(X^2) < \infty$,

(ii) $W_n(1/n)f(x) = f(x)$ (*each* $f \in C(J)$; $n \in \mathbb{N}$), i.e., $W_n(1/n)$ *interpolates at* x.

Proof. To utilize the theorems given above for $\varphi(n) = n^{-1}$, we have to determine the limiting r.v. Z and the associated components. For fixed $x \in \mathbb{R}$ let Z be distributed as X_x, i.e.,

$$F_Z(u) = F_{X_x}(u) := 0, \qquad -\infty < u < x$$
$$:= 1, \qquad x \leq u < \infty.$$

In other words, $P(\{X_x = x\}) = 1$. Notice that here the limiting r.v. Z also depends on $x \in J$. It is easy to see that X_x is $1/n$-decomposable with components $Z_{i,n}$ all distributed as X_x; hence

$$F_{X_x} = F_{(1/n)\Sigma_{i=1}^n Z_{i,n}}.$$

For the moments of X_x we have $E(X_x{}^r) = x^r$. In view of (1.6) and

$$\int_{-\infty}^{\infty} f(u)\, dF_{X_x}(u) = f(x) \tag{3.5}$$

(3.2) can therefore be written in the form

$$\int_{-\infty}^{\infty} f(u)\, d[F_{S_n/n}(u) - F_{X_x}(u)] = o(1) \qquad (\text{each } f \in C_b{}^1(J);\ n \to \infty).$$

To prove part (a), use Theorem B with $r = 1$ to yield $E(X) < \infty$ and $E(X) = E(X_x) = x$. This leads to

$$W_n(1/n) f_1(x) = E(S_n/n) = \frac{1}{n} \sum_{i=1}^{n} E(X) = x.$$

Concerning part (b), Theorem A is applicable with $r = 2$, $\beta = 1 + \alpha$, so that $[\min(r, \beta)] = 2$.

Finally, for part (c) use Theorem B again, but now for $r = 2$, so that $E(X^2) < \infty$ follows immediately, and additionally

$$E(X^j) = E(X_x{}^j) \qquad (0 \leqslant j \leqslant 2).$$

Hence $E(X) = x$ and $E(X^2) = x^2$. Then

$$\operatorname{Var}(X) := E([X - E(X)]^2) = E(X^2) - x^2 = 0, \tag{3.6}$$

leading to $X = x$ P-a.s. Therefore $F_X = F_{X_x}$, which implies $F_{S_n/n} = F_{X_x}$. Finally,

$$\begin{aligned} W(1/n) f(x) &= \int_{-\infty}^{\infty} f(u)\, dF_{S_n/n}(u) \\ &= \int_{-\infty}^{\infty} f(u)\, dF_{X_x}(u) = f(x), \end{aligned} \tag{3.7}$$

even for each measurable and bounded f since $W_n(1/n)$ is defined for such f. This completes the proof of Theorem 1.

It seems worthwhile to combine the inverse Theorem 1 with direct theorems of [7]. So one has the following results for $\mathfrak{W}(1/n)$, the set of sequences of operators of the form (1.6) with $\varphi(n) = n^{-1}$ and $x \in J$. Below, for example, $\{|W_n(1/n)f(x) - f(x)| = o(1),\ f \in C_b^1(J)\}$ means the set of all sequences $(W_n(1/n))_{n=1}^\infty \in \mathfrak{W}(1/n)$ such that $|W_n(1/n)f(x) - f(x)| = o(1)$, $n \to \infty$, for each $f \in C_b^1(J)$.

THEOREM 2. *Let $W_n(1/n)f(x)$ be defined by* (3.1). *Then one has*

(a) $\{|W_n(1/n)f(x) - f(x)| = o(1), f \in C_b^1(J)\} = \{W_n(1/n)f_1(x) = f_1(x),$ $n \in \mathbb{N}$, *and* $E(X) < \infty\}$,

(b) $\{|W_n(1/n)f(x) - f(x)| = \mathcal{O}(n^{-1}), \quad f \in C_b^2(J)\} = \{W_n(1/n)f_1(x) =$ $f_1(x)$, $n \in \mathbb{N}$, *and* $E(X^2) < \infty\}$,

(c) $\{|W_n(1/n)f(x) - f(x)| = o(n^{-1}), f \in C_b^2(J)\} = \{W_n(1/n)f(x) = f(x),$ $n \in \mathbb{N}, f \in C_b(J)$, *and* $E(X^2) < \infty\}$.

To discuss assertions (a), (b) and (c), if one assumes directly that $E(X^2) < \infty$, i.e., $C_b^2(J)$ belongs to the domain of $W_n(1/n)$, $n \in \mathbb{N}$, then a comparison of (a) and (b) shows that a pointwise approximation process of $\mathfrak{W}(1/n)$ automatically approximates $f(x)$ with $\mathcal{O}(1/n)$ (for smooth f). Moreover, (b) and (c) together show that $\mathfrak{W}(1/n)$ is satiated with satiety order $1/n$, i.e., the rate $o(1/n)$ is impossible, unless $W_n(1/n)$ is the identity I.

Notice that the last assertion gives a partial answer to a problem posed by P. L. Butzer in 1963 and recalled by a number of authors (compare, e.g., [1, 5]): Is it possible to construct a sequence of algebraic polynomials, defined on [0, 1], which are of the same structure as the Bernstein polynomials and which approximate the associated function f with an order $\mathcal{O}(n^{-2})$ on [0, 1] provided $f'' \in C[0, 1]$? As mentioned above, the sequence of the Bernstein operators belongs to $\mathfrak{W}(1/n)$, and Theorem 2 states in this respect that as long as one modifies the Bernstein operators in such a fashion that the new operators still belong to $\mathfrak{W}(1/n)$, then a better rate than $\mathcal{O}(n^{-1})$ is impossible. So in this case the answer to the problem just mentioned is negative.

Recalling the well-known Bohman–Korovkin theorem, we know that for the class $\mathfrak{U}$ the (test) functions f_0, f_1 and f_2 already guarantee that $(U_n)_{n=1}^\infty \in \mathfrak{U}$ is an approximation process. If one now considers the smaller class $\mathfrak{W}(1/n) \subset \mathfrak{U}$, then (a) shows that the two functions f_0, f_1 already guarantee $(W_n(1/n))_{n=1}^\infty \in \mathfrak{U}(1/n)$ to be an approximation process. But (a) gives even more information. If one has a sequence of positive linear operators U_n that defines an approximation process, and $U_n f_1(x) = f_1(x)$ does not hold (but of course approximatively for $n \to \infty$), then U_n cannot have the probabilistic form (3.1), i.e., there cannot exist r.v. X_i such that $U_n f(x) = E(f \circ [S_n/n])$.

4. Results for the Limiting U Connected with the Normal Distribution

For a second application of Theorems A and B we choose $\varphi(n) = 1/\sqrt{n}$ and $F_Z = F_{X^*}$, where X^* is a normally distributed r.v. with mean 0 and variance 1, i.e.,

$$F_{X^*}(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

Concerning (1.6) we now consider operators $(W_n(1/\sqrt{n}))_{n=1}^{\infty} \in \mathfrak{W}(1/\sqrt{n})$ of the form

$$W_n(1/\sqrt{n}) f(x) = E(f \circ [S_n/\sqrt{n}]) \qquad (f \in K), \tag{4.1}$$

the limiting operator now being

$$Uf = U_1 f := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-t^2/2}\, dt, \tag{4.2}$$

which is independent of $x \in \mathbb{R}$. Then we have the following theorem (the proof follows from [6])

THEOREM 3. *Let $x \in R$ be fixed.*

(a) *If*

$$| W_n(1/\sqrt{n}) f(x) - U_1 f | = \mathcal{O}(n^{-\alpha}) \qquad (\textit{each } f \in C_b^{]2\alpha+2[}(\mathbb{R})) \tag{4.3}$$

for some $\alpha > 0$, then

$$\begin{aligned} &\text{(i)} \quad E(X^{]2\alpha+2[}) < \infty, \\ &\text{(ii)} \quad E(X^j) = E(X^{*j}) \qquad (0 \leqslant j \leqslant [2\alpha + 2] - 1). \end{aligned} \tag{4.4}$$

(b) *If*

$$| W_n(1/\sqrt{n}) f(x) - U_1 f | = o(n^{-r/2}) \qquad (\textit{each } f \in C_b^{r+2}(\mathbb{R});\ n \to \infty) \tag{4.5}$$

for some $r \in \mathbb{P}$, then

$$\begin{aligned} &\text{(i)} \quad E(X^{r+2}) < \infty, \\ &\text{(ii)} \quad E(X^j) = E(X^{*j}) \qquad (0 \leqslant j \leqslant r + 2). \end{aligned} \tag{4.6}$$

With corresponding direct results of [4], conditions (4.4) and (4.6) also imply (4.3) and (4.5), respectively, the counterpart of Theorem 2 could therefore be formulated, characterizing the operator classes $\mathfrak{W}_\alpha(1/\sqrt{n}) \subset \mathfrak{W}(1/\sqrt{n})$.

In contrast to the previous case (with the limiting operator I), $\mathfrak{W}(1/\sqrt{n})$ is not polynomially satiated since each fixed arbitrary high order of approximation $n^{-\alpha}$, $\alpha > 0$, of $(W_n(1/\sqrt{n}))_{n=1}^{\infty} \in \mathfrak{W}(1/\sqrt{n})$ towards U_1 is possible (cf. (4.4)) without implying $F_X = F_{X^*}$, or equivalently $W_n(1/\sqrt{n}) = U_1$, $n \in \mathbb{N}$. But if one would require exponential approximation order e^{-an}, some $a > 0$, implying

$$\| W_n(1/\sqrt{n}) f(x) - U_1 f \| = \mathcal{O}(n^{-\alpha})$$

for each $\alpha > 0$, then (4.4) would yield that all moments of the r.v. X coincide with those of X^*, thus $F_X = F_{X^*}$, since the moments of X^* are known to generate a well-defined Hamburger moment problem.

On the other hand, given an arbitrary sequence of r.v. X_i that determine a corresponding operator, one may transform this sequence by setting $X_i' := [X_i - E(X_i)]/(\mathrm{Var}(X_i))^{1/2}$ if $0 < \mathrm{Var}(X_i) < \infty$, so that $E(X^j) = E(X^{*j})$, $j = 0, 1, 2$. [Of course, in general $E(X^3) \neq E(X^{*3})$ $(= 0)$]. This means that for these operators the best possible rate of convergence is given by

$$\| W_n(1/\sqrt{n}) f(x) - U_1 f \| = \mathcal{O}(n^{-1/2}) \qquad (f \in C_b^3(\mathbb{R})).$$

This is the situation for binomially and exponentially distributed r.v. X leading to Bernstein and Szàsz–Mirakjan operators. An example of r.v. which yield higher order of approximation is given in [2]. For each $x \in (0, \frac{1}{2}]$, the corresponding r.v. $X(x)$ are defined via their distributions by $P(\{X(x) = -(2x)^{-1/2}\}) = P(\{X(x) = (2x)^{-1/2}\}) = x$ and $P(\{X(x) = 0\}) = 1 - 2x$. Some calculations yield that the corresponding operator $B_n(1/\sqrt{n})f(x) = E(f \circ [S_n/\sqrt{n}])$, $x \in (0, \frac{1}{2}]$, is of the form

$$B_n(1/\sqrt{n}) f(x) = \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} x^k (1 - 2x)^{n-k} f\left(\frac{k - 2j}{\sqrt{2nx}}\right).$$

For this operator, which is a polynomial of degree n, we have as best possible rate for $x \in (0, \frac{1}{2}] \setminus \{\frac{1}{6}\}$

$$\| B_n(1/\sqrt{n}) f(x) - U_1 f \| = \mathcal{O}(n^{-1}) \qquad (\text{each } f \in C_b^4(\mathbb{R}))$$

and for $x = \frac{1}{6}$

$$\| B_n(1/\sqrt{n}) f(\tfrac{1}{6}) - U_1 f \| = \mathcal{O}(n^{-2}) \qquad (\text{each } f \in C_b^6(\mathbb{R})).$$

Let us conclude this paper with the following remark: An important result of A. Y. Khintchine (cf., e.g., [8]) states that if $\varphi(n) S_n$ converges in distribution to a r.v. Z, then Z must be stably distributed; convergence in distribution means that

$$\lim_{n\to\infty} E(f\circ[\varphi(n)\,S_n]) = \lim_{n\to\infty} W_n(\varphi(n))\,f(x)$$

$$= E(f\circ Z) \qquad (\text{each } f\in C_b^r(\mathbb{R}))$$

for any $r\in\mathbb{P}$. Thus all possible limiting operators U of sequences of operators of probabilistic type (1.6) are determined by the stably distributed r.v. Z, namely $Uf = E(f\circ Z)$. If one now assumes additionally that $f_2\in D(U)$ —which is natural when considering rates of convergence—or, equivalently that the second moment of Z is finite, then the only remaining stably distributed r.v. Z are X_x and X^*. Furthermore, the φ-decomposability of these limiting r.v. is needed, which is exactly true for $\varphi(n) = n^{-1}$ and $\varphi(n) = n^{-1/2}$, respectively. This is the reason why the probabilistic approach of this paper only allows one to consider the operators $W_n(1/n)$, $W_n(1/\sqrt{n})$ with the corresponding limiting operators I and U_1.

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